Nonconvex Rigid Bodies with Stacking Implementation Notes

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1 Notation

After applying an impulse $\boldsymbol{j}$ at a point $\boldsymbol{r}$ the new linear and angular velocities will be

$$\boldsymbol{v}' = \boldsymbol{v} + \frac{\boldsymbol{j}}{m}$$

$$\boldsymbol{\omega}' = \boldsymbol{\omega} + \boldsymbol{I}^{-1}(\boldsymbol{r} \times \boldsymbol{j})$$

It is possible to transform the second equation using the cross product matrix of $\boldsymbol{r}$

$$\boldsymbol{r}_* = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

This matrix has the property that, for a vector $\boldsymbol{b}$

$$\boldsymbol{r} \times \boldsymbol{b} = \boldsymbol{r}_* \boldsymbol{b}$$

$$\boldsymbol{b} \times \boldsymbol{r} = \boldsymbol{r}_*{}^T \boldsymbol{b}$$

This gives:

$$\boldsymbol{\omega}' = \boldsymbol{\omega} + \boldsymbol{I}^{-1}(\boldsymbol{r}_* \boldsymbol{j})$$

The instantaneous velocity $\boldsymbol{u}$ at the point $\boldsymbol{r}$ is

$$\boldsymbol{u} = \boldsymbol{v} + \omega \times \boldsymbol{r}$$

After the impulse is applied, the instantaneous velocity will be

$$\boldsymbol{u} = \boldsymbol{v}' + \omega' \times \boldsymbol{r}$$

$$\boldsymbol{u}' = \boldsymbol{v} + \frac{\boldsymbol{j}}{m} + \left(\omega + \boldsymbol{I}^{-1}(\boldsymbol{r}_* \boldsymbol{j})\right) \times \boldsymbol{r}$$

Which can be rewritten in terms of $\boldsymbol{u}$ and $\boldsymbol{j}$

$$
\begin{aligned}
\boldsymbol{u}' &= \boldsymbol{v} + \omega \times \boldsymbol{r} + \frac{\boldsymbol{j}}{m} + \left(\boldsymbol{I}^{-1}\left(\boldsymbol{r}_{*}\boldsymbol{j}\right)\right) \times \boldsymbol{r} \\
\boldsymbol{u}' &= \boldsymbol{u} + \frac{\boldsymbol{j}}{m} + \boldsymbol{r}_{*}{}^{T}\left(\boldsymbol{I}^{-1}\left(\boldsymbol{r}_{*}\boldsymbol{j}\right)\right) \\
\boldsymbol{u}' &= \boldsymbol{u} + \left(\frac{\boldsymbol{\delta}}{m} + \boldsymbol{r}_{*}{}^{T}\boldsymbol{I}^{-1}\boldsymbol{r}_{*}\right)\boldsymbol{j} \\
\boldsymbol{u}' &= \boldsymbol{u} + \boldsymbol{K}\boldsymbol{j}
\end{aligned}
$$

where $\boldsymbol{\delta}$ is the 3 by 3 identity matrix.

When bodies a and b collide, the impulse $\boldsymbol{j}$ is applied in an equal and opposite manner to each body. The relative velocity at the point $\boldsymbol{r}$ (expressed in each body's coordinate system) will be

$$
\begin{aligned}
\boldsymbol{u}'_{rel} &= \boldsymbol{u}'_{a} - \boldsymbol{u}'_{b} \\
\boldsymbol{u}'_{rel} &= \left(\boldsymbol{u}_{a} + \boldsymbol{K}_{a}\boldsymbol{j}\right) - \left(\boldsymbol{u}_{b} + \boldsymbol{K}_{b}\left(-\boldsymbol{j}\right)\right) \\
\boldsymbol{u}'_{rel} &= \boldsymbol{u}_{a} - \boldsymbol{u}_{b} + \left(\boldsymbol{K}_{a} + \boldsymbol{K}_{b}\right)\boldsymbol{j} \\
\boldsymbol{u}'_{rel} &= \boldsymbol{u}_{rel} + \boldsymbol{K}_{T}\boldsymbol{j}
\end{aligned}
$$

From now on the subscript of the relative velocity $\boldsymbol{u}_{rel}$ will be dropped for simplicity.

2 Frictionless impulse

The relative velocity $\boldsymbol{u}$ can be split into a normal component $u_n\boldsymbol{n}$ along the normal $\boldsymbol{n}$ to the collision and a tangential component $u_t\boldsymbol{t} = \boldsymbol{u} - u_n\boldsymbol{n}$ in the plane of the collision, where $\boldsymbol{n}$ and $\boldsymbol{t}$ are normalized. The same can be done for j. After the impulse is applied the normal component of the relative velocity at the point $\boldsymbol{r}$ will be

$$
\begin{aligned}
u'_n &= \left(u + \boldsymbol{K}_{T}\boldsymbol{j}\right) \cdot \boldsymbol{n} \\
u'_n &= u_n + \boldsymbol{n}^{T}\left(\boldsymbol{K}_{T}\boldsymbol{j}\right) \\
u'_n &= u_n + \boldsymbol{n}^{T}\boldsymbol{K}_{T}\boldsymbol{n}j_n
\end{aligned}
$$

The purpose of the impulse $\boldsymbol{j}$ is to reverse the velocity and scale it by the coefficient of restitution ϵ along the normal $\boldsymbol{n}$, that is $u'_n = -\epsilon u_n$. The impulse is therefore purely normal ($\boldsymbol{j} = j_n\boldsymbol{n}$). Substituting into the previous equation and solving for j_n gives

$$
j_n = \frac{-\left(1+\epsilon\right)u_n}{\boldsymbol{n}^{T}\boldsymbol{K}_{T}\boldsymbol{n}}
$$

3 Frictional impulse

In addition to the previous impulse, assume static friction, which means the objects must not slide along the tangential direction, i.e. $u_t = 0$. This gives $\boldsymbol{u}' = -\epsilon u_n \boldsymbol{n}$, or

$$-\epsilon u_n \boldsymbol{n} = \boldsymbol{u} + \boldsymbol{K}_T \boldsymbol{j}$$

which can be solved for $\boldsymbol{j}$ to get

$$\boldsymbol{j} = \boldsymbol{K}_T^{-1} \left(-\boldsymbol{u} - \epsilon u_n \boldsymbol{n} \right)$$

Given a coefficient of friction μ the maximum impulse that the frictional force can apply to resist motion is μj_n. Therefore switch to kinetic friction instead if $j_t \geq \mu j_n$. The frictional impulse in that case will be $\boldsymbol{j} = j_n \boldsymbol{n} - \mu j_n \boldsymbol{t} = j_n \boldsymbol{k}$. Subsitute and solve for j_n

$$\begin{aligned} -\epsilon u_n &= u_n + \boldsymbol{n}^T \boldsymbol{K}_T \boldsymbol{k} j_n \\ j_n &= \frac{-\left(1+\epsilon\right) u_n}{\boldsymbol{n}^T \boldsymbol{K}_T \boldsymbol{k}} \end{aligned}$$

4 2D case

In the 2D case, by ignoring the z component of $\boldsymbol{r}_*$ we get

$$\boldsymbol{K} = \frac{\boldsymbol{\delta}}{m} + I^{-1} \left[\begin{array}{cc} r_y^2 & -r_x r_y \\ -r_x r_y & r_x^2 \end{array} \right]$$

where $\boldsymbol{\delta}$ is the 2 by 2 identity matrix. Everything else stays the same.